A) <u>ALGEBRAIC PROOFS</u>

Reminder: Represent the complex numbers in rectangular form if they have not been readily defined yet.

Example: Show that $\overline{w}z = \overline{w\overline{z}}$ if w and z are two complex numbers.

Let $w = x + yi$ and $z = a + bi$. Therefore $\overline{w} = x - yi$ and $\overline{z} = a - bi$.

$$\begin{aligned} \text{LHS} = \overline{w}z = (x - yi)(a + bi) &= x \cdot a + x \cdot bi - a \cdot yi - by \cdot i^2 \\ &= ax + bxi - ayi + by \\ &= (ax + by) + (bx - ay)i \end{aligned}$$

reminder: $i^2 = -1$

$$\begin{aligned} \text{RHS} = \overline{w\overline{z}} = \overline{(x + yi)(a - bi)} &= \overline{x \cdot a - x \cdot bi + a \cdot yi - by \cdot i^2} \\ &= \overline{ax - bxi + ayi + by} \\ &= \overline{(ax + by) - (bx - ay)i} \\ &= (ax + by) + (bx - ay)i \\ &= \text{LHS} \end{aligned}$$

Therefore $\overline{w}z = \overline{w\overline{z}}$

B) <u>LOCI</u>

Reminder: Represent the complex numbers in rectangular form if they have not been readily defined yet.

Example: Find the locus (set of points) of z if $|z - 1| = 1$.

Let $z = x + yi$. Then
$$\begin{aligned} |z - 1| &= |x + yi - 1| \\ &= |(x - 1) + yi| \\ &= \sqrt{(x - 1)^2 + y^2} \end{aligned}$$

reminder: $|a + bi| = \sqrt{a^2 + b^2}$

Therefore $|z - 1| = 1$
can be written as $\sqrt{(x - 1)^2 + y^2} = 1$.

Squaring both sides leaves you with $(x - 1)^2 + y^2 = 1$.
This is the equation for a circle of radius 1 and centre (1, 0).
This is the locus (set of points) for z.

C) MULTI-STEP EQUATIONS

Reminder: A polynomial of degree n has exactly n roots.

Example: Completely solve the following equation: $(z^2+2)^3 = 1$.

$(z^2+2)^3 = 1$

$(W)^3 = 1$ *let $W = z^2+2$*

$W^3 = 1\ cis\ 0°$

$W^3 = 1\ cis\ (0° + 360°n)$

$W^3 = 1\ cis\ (360°n)$

$W = [1\ cis\ (360°n)]^{1/3}$

$W = 1\ cis\ (120°n)$ *De Moivre's Theorem*

$n = 0: W_1 = 1\ cis\ 0°$

$n = 1: W_2 = 1\ cis\ 120°$

$n = 2: W_3 = 1\ cis\ 240° = 1\ cis\ (-120°)$

3 solutions, as we're solving W^3 i.e. a cubic

Since $W = z^2+2$, then $z^2+2 = W_1$, $z^2+2 = W_2$, and $z^2+2 = W_3$.

Solving $z^2+2 = W_1$:

$z^2+2 = 1\ cis\ 0° = 1$

$z^2 = 1-2$

$z^2 = -1 = 1 \text{ cis } 180°$

$z^2 = 1 \text{ cis}(180° + 360°n)$

$z = [1 \text{ cis}(180° + 360°n)]^{1/2}$

$z = 1 \text{ cis}(90° + 180°\text{n})$ *De Moivre's Theorem*

$n = 0: z_1 = 1 \text{ cis } 90°$

$n = 1: z_2 = 1 \text{ cis } 270° = 1 \text{ cis}(-90°)$

Solving $z^2+2 = W_2$:

$z^2+2 = 1\ cis\ 120°$

$z^2+2 = \cos 120° + i \sin 120°$

$z^2 = \cos 120° + i \sin 120° - 2$

$z^2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i - 2$

$z^2 = -\frac{5}{2} + \frac{\sqrt{3}}{2}i$

$z^2 = \sqrt{7} \text{ cis}(160.9°)$ *change to polar form*

$z = [\sqrt{7} \text{ cis}(160.9° + 360°\text{n})]^{1/2}$

$z = 1.63 \text{ cis}(80.45° + 180°\text{n})$ *De Moivre's Theorem*

$n = 0: z_3 = 1.63 \text{ cis } (80.45°)$

$n = 1: z_4 = 1.63 \text{ cis } 260.45° = 1.63 \text{ cis}(-99.55°)$

Solving $z^2 + 2 = W_3$:

$$z^2 + 2 = 1\, cis\, (-120°)$$
$$z^2 + 2 = \cos(-120°) + i \sin(-120°)$$
$$z^2 = \cos(-120°) + i \sin(-120°) - 2$$
$$z^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2} i - 2$$
$$z^2 = -\frac{5}{2} - \frac{\sqrt{3}}{2} i$$
$$z^2 = \sqrt{7}\ \mathrm{cis}(-160.9°)$$ *change to polar form*
$$z = [\sqrt{7}\ \mathrm{cis}(-160.9° + 360°\mathrm{n})]^{1/2}$$
$$z = 1.63\ \mathrm{cis}(-80.45° + 180°\mathrm{n})$$ *De Moivre's Theorem*

$n = 0$: $z_5 = 1.63\ \mathrm{cis}\ (-80.45°)$

$n = 1$: $z_6 = 1.63\ \mathrm{cis}\ (99.55°)$

So the solutions are: $z_1 = 1\ \mathrm{cis}\ 90°$, $z_2 = 1\ \mathrm{cis}(-90°)$, $z_3 = 1.63\ \mathrm{cis}\ (80.45°)$, $z_4 = 1.63\ \mathrm{cis}(-99.55°)$, $z_5 = 1.63\ \mathrm{cis}\ (-80.45°)$, and $z_6 = 1.63\ \mathrm{cis}\ (99.55°)$.